Stability of a Viscoelastic Rotor-Disk System Under Dynamic Axial Loads

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Introduction

BECAUSE of increasing use of newer composites and polymers in aerospace technology, considerable research interest[1-3] is being focused on analyzing the dynamic characteristics of components made up of viscoelastic materials. Whirling instability caused by the presence of internal damping in a rotor shaft[4-8] has been a matter of concern for some time. However, in the majority of analytical solutions, it is common practice to assume a simple shaft with either no attached disks or no external forces at all. In the present work, we consider the stability problem of an axially loaded uniform viscoelastic rotor with multiple disks mounted at discrete locations. Starting from the basic beam equations along with the rotary iner-tia and gyroscopic effect terms, a complete set of coupled Mathieu-Hill equations have been derived for a rotor supported on short rigid bearings. The harmonic balance method of dynamic stability analysis is recalled here only very briefly in order to keep this Note self-contained; for a more detailed discussion of this topic, the reader should refer to the author's previous work.[9]

Mathematical Derivation

In this analysis the disks are considered relatively thin and rigid with respect to the shaft. A total number of n disks are mounted on the shaft at different locations, commonly referred to as stations in the usual Mykelstad-Prohl technique. In our mathematical derivation they are handled by a Dirac's delta impulse function defined as

$$\int_0^\ell \delta(z-z_j)\,\mathrm{d}z = 1$$

where the $\delta(z-z_j)$ function has the dimension of $(\text{length})^{-1}$ and

$$\delta(z-z_j) = 0 \qquad \text{for} \qquad z \neq z_j$$

Also, for any arbitrary function of z, say $\phi(z)$, the definite integral is

$$\int_0^\ell \delta(z-z_j)\phi(z)\,\mathrm{d}z = \phi(z_j)$$

The accuracy of using the Dirac's impulse function in the dynamic analysis of point masses attached to a continuum structure has been shown earlier by this author.[10] At this point, along with the legends used in Fig. 1, we also introduce the following notations for equivalent density and inertias of the shaft per unit length:

Equivalent shaft material density:

$$\rho_{\text{eq}} = \rho + \frac{1}{A}\sum_{j=1}^{n} M_j\delta(z-z_j)$$

Received Dec. 22, 1988; revision received Feb. 25, 1989.

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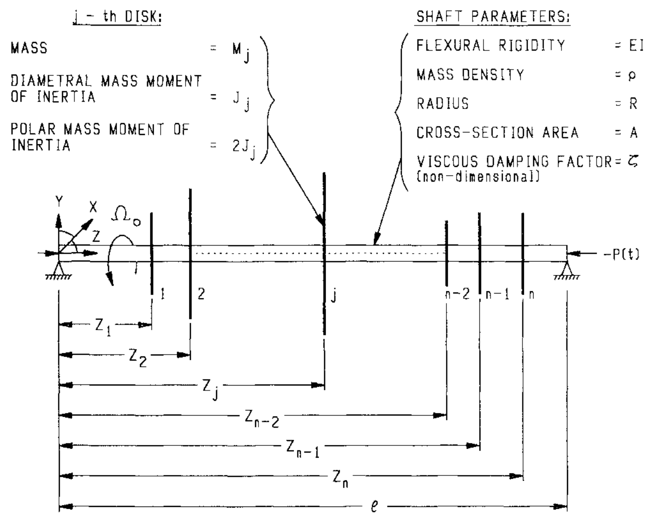

Fig. 1 Schematic diagram of the rotor shaft with multiple disks.

Equivalent diametral mass moment of inertia:

$$I_d = \frac{\rho A R^2}{4} + \sum_{j=1}^{n} J_j\delta(z-z_j)$$

Equivalent polar mass moment of inertia:

$$I_p = 2I_d$$

Critical viscous damping coefficient of the shaft:

$$D = \tfrac{1}{2}\pi^2\ell^2[EI\rho A]^{1/2}$$

Using these notations, the equations of motion for the lateral deflection (u,v) in the x-y plane of a uniform cross section rotor $(0 \le z \le \ell)$, shown in Fig. 1, under combined constant and periodic axial loads P_o and P_t are written as[11-13]

$$\mathcal{L}_1(u,v) = \left[EI + 2\zeta D\frac{\partial}{\partial t}\right]u_{,zzzz} + \rho_{\text{eq}}Au_{,tt} - I_d u_{,zztt} - I_p\Omega_0 v_{,zzt} - (P_o + P_t\cos\Omega t)u_{,zz} = 0 \tag{1}$$

$$\mathcal{L}_2(u,v) = \left[EI + 2\zeta D\frac{\partial}{\partial t}\right]v_{,zzzz} + \rho_{\text{eq}}Av_{,tt} - I_d v_{,zztt} + I_p\Omega_0 u_{,zzt} - (P_o + P_t\cos\Omega t)v_{,zz} = 0 \tag{2}$$

where

EI = flexural rigidity of the rotor
P_o = constant axial force component
P_t = periodic axial force component
R = radius of the shaft
ζ = internal damping factor due to material viscoelasticity (nondimensional
Ω = frequency of axial force P_t, rad/s
Ω_0 = angular velocity of rotation

The preceding equations assume that the internal damping in the shaft is proportional to the velocity of each longitudinal fiber, and also that the positive and negative signs for the axial force P indicate tension and compression, respectively. It is worth noting that in the majority of rotating system applications, the excitation frequency Ω is usually an integral multiple of the rotational velocity Ω_0. However, to keep the derivations general, we will treat them as independent quantities. In this discussion we are considering the example of a rotor with short bearings at the two ends, where the end conditions can be modeled as simply supported. In simply supported end conditions, the solution of Eqs. (1) and (2) may be assumed to be a product of the function of time t and the spatial variable z, i.e.,

$$u(z,t) = \sum_{k=1}^{\infty} [f_1(t)]_k \sin\left(\frac{k\pi z}{\ell}\right) \tag{3}$$

$$v(z,t) = \sum_{k=1}^{\infty} [f_2(t)]_k \sin\left(\frac{k\pi z}{\ell}\right) \tag{4}$$

where f_1 and f_2 are complex functions of time t, and k represents the buckled mode shape number. The coefficients containing the time variable are complex in nature due to the presence of viscous damping and a gyroscopic effect in the system. From the expressions in Eqs. (3) and (4) and applying Galerkin's method, one obtains a set of ordinary differential equations of which a typical kth equation is

$$\int_0^{\ell} \mathcal{L}(u,v) \sin\left(\frac{k\pi z}{\ell}\right) \mathrm{d}z = 0 \tag{5}$$

Here, $\mathcal{L}$ denotes the differential operator introduced earlier in Eqs. (1) and (2). For any practical situation, one would be interested only in the first buckled mode, i.e., $k=1$. This step reduces the partial differential equations, Eqs. (1) and (2), to a set of two ordinary differential equations with complex functions of time t. In deriving the preceding equations, we make use of the following orthogonal relations:

$$\int_0^{\ell} \sin\left(\frac{k\pi z}{\ell}\right) \sin\left(\frac{k\pi z}{\ell}\right) \mathrm{d}z = \ell/2$$

$$\int_0^{\ell} \delta(z-z_j) \sin\left(\frac{k\pi z}{\ell}\right) \sin\left(\frac{k\pi z}{\ell}\right) \mathrm{d}z = ½\left[1-\cos\frac{2k\pi z_j}{\ell}\right]$$

Thus, one finally ends up with a set of two coupled ordinary differential equations. For the sake of brevity, the new set of equations can be written in matrix form as

$$\boldsymbol{M}\{\ddot{f}\} + \boldsymbol{C}\{\dot{f}\} + (\boldsymbol{K} - \boldsymbol{B}\cos\Omega t)\{f\} = \{0\} \tag{6}$$

where $\{f\}^T = \{f_1 f_2\}$ and $(\dot{\ })$ denotes ordinary derivative with respect to time t. The nonzero terms of the matrices $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ and $\boldsymbol{B}$ are

$$M_{1,1} = M_{2,2} = \frac{\rho A \ell}{2}\left(1 + \frac{\pi^2 R^2}{4\ell^2}\right) + \frac{1}{2}\sum_{j=1}^{n}\left(M_j + \frac{\pi^2}{\ell^2}J_j\right)\left(1-\cos\frac{2\pi z_j}{\ell}\right)$$

$$C_{1,1} = C_{2,2} = \zeta\frac{\pi^6}{2\ell}[EI\rho A]^{½}$$

$$C_{1,2} = -C_{2,1} = \Omega_0\frac{\pi^2}{\ell^2}\left[\frac{\rho A R^2 \ell}{4} + \sum_{j=1}^{n} J_j\left(1-\cos\frac{2\pi z_j}{\ell}\right)\right]$$

$$K_{1,1} = K_{2,2} = EI\frac{\pi^4}{2\ell^3} + P_0\frac{\pi^2}{2\ell}$$

$$B_{1,1} = B_{2,2} = -P_t\frac{\pi^2}{2\ell}$$

In the preceding relations, the reader is reminded that M_j stands for the mass of the jth disk, whereas $\boldsymbol{M}$ with double subscripts denotes the elements of the $\boldsymbol{M}$ matrix. Here, it should be noted that while $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{B}$ are symmetric matrices, the matrix $\boldsymbol{C}$ is skew-symmetric. Equation (6) can be identified as a set of modified coupled Mathieu-Hill equations with additional contributions from the velocity-dependent coefficient matrix $\boldsymbol{C}$. As outlined in the author's previous work,[9] the solution of Eq. (6) here is sought in the form of a product of two functions: the first is an exponential function with characteristic exponent λ, and the second is a periodic function. The periodic function can be broken into two components: one made up of a time period equal to twice that of the axial forcing term (i.e., $4\pi/\Omega$) as

$$\{f(t)\} = e^{\lambda t}\left[\sum_{m=1,3,5\ldots}^{\infty}\left(\{a_m\}\sin\frac{m\Omega t}{2} + \{b_m\}\cos\frac{m\Omega t}{2}\right)\right] \tag{7a}$$

and the other equal to the period of the forcing function (i.e., $2\pi/\Omega$) as

$$\{f(t)\} = e^{\lambda t}\left[\frac{\{b_o\}}{2} + \sum_{m=2,4,6,8\ldots}^{\infty}\left(\{a_m\}\sin\frac{m\Omega t}{2} + \{b_m\}\cos\frac{m\Omega t}{2}\right)\right] \tag{7b}$$

where $\{a_m\}$ and $\{b_m\}$ are time-independent Fourier coefficients. Substitution of the series representation of $\{f(t)\}$ shown in Eqs. (7) enables us to use the method of harmonic balance.[11] In this scheme we collect the coefficients of $e^{\lambda t}$, $e^{\lambda t}\sin(m\Omega t/2)$, and $e^{\lambda t}\cos(m\Omega t/2)$, respectively, and equate them to zero. This yields the following set of homogeneous algebraic equations:

$$(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K})\{b_0\} - \boldsymbol{B}\{b_2\} = \{0\} \tag{8}$$

$$\lambda^2\boldsymbol{M}\{a_1\} - \lambda\left(\Omega\boldsymbol{M}\{b_1\} - \boldsymbol{C}\{a_1\}\right) + \left(\boldsymbol{K} - \frac{\Omega^2}{4}\boldsymbol{M}\right)\{a_1\} - \frac{\Omega}{2}\boldsymbol{C}\{b_1\} - \frac{1}{2}\boldsymbol{B}\{a_3\} = \{0\} \tag{9}$$

$$\lambda^2\boldsymbol{M}\{b_1\} + \lambda\left(\Omega\boldsymbol{M}\{a_1\} + \boldsymbol{C}\{b_1\}\right) + \left(\boldsymbol{K} - \frac{\Omega^2}{4}\boldsymbol{M}\right)\{b_1\} + \frac{\Omega}{2}\boldsymbol{C}\{a_1\} - \frac{1}{2}\boldsymbol{B}\{b_3\} = \{0\} \tag{10}$$

$$\lambda^2\boldsymbol{M}\{a_2\} - \lambda\left(2\Omega\boldsymbol{M}\{b_2\} - \boldsymbol{C}\{a_2\}\right) + \left(\boldsymbol{K} - \Omega^2\boldsymbol{M}\right)\{a_2\} - \Omega\boldsymbol{C}\{b_2\} - \frac{1}{2}\boldsymbol{B}\{a_4\} = \{0\} \tag{11}$$

$$\lambda^2\boldsymbol{M}\{b_m\} + \lambda\left(m\Omega\boldsymbol{M}\{a_m\} + \boldsymbol{C}\{b_m\}\right) + \left(\boldsymbol{K} - \frac{m^2\Omega^2}{4}\boldsymbol{M}\right)\{b_m\} + \frac{m\Omega}{2}\boldsymbol{C}\{a_m\} - \frac{1}{2}\boldsymbol{B}\left(\{b_{m-2}\} + \{b_{m+2}\}\right) = \{0\} \tag{12}$$

where $m = 2,3,4,5\ldots$ and

$$\lambda^2\boldsymbol{M}\{a_m\} - \lambda\left(m\Omega\boldsymbol{M}\{b_m\} - \boldsymbol{C}\{a_m\}\right) + \left(\boldsymbol{K} - \frac{m^2\Omega^2}{4}\boldsymbol{M}\right)\{a_m\} - \frac{m\Omega}{2}\boldsymbol{C}\{b_m\} - \frac{1}{2}\boldsymbol{B}\left(\{a_{m-2}\} + \{a_{m+2}\}\right) = \{0\} \tag{13}$$

where $m = 3,4,5,6\ldots$.

These systems of equations can be rearranged with three different matrices, viz. $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$ as the coefficients of ascending powers of λ in the following form:

$$(\boldsymbol{P} + \lambda\boldsymbol{Q} + \lambda^2\boldsymbol{R})\{X\} = \{0\} \tag{14}$$

where the column vector $\{X\}$ contains the unknown Fourier coefficients.

The eigenvalue problem of Eq. (14) can be solved easily by first converting the equation into a "state" form by introducing a new column vector $\{Y\} = \lambda\{X\}$ as

$$\begin{bmatrix} 0 & I \\ -R^{-1}P & -R^{-1}Q \end{bmatrix} \begin{Bmatrix} \{X\} \\ \{Y\} \end{Bmatrix} = \lambda \begin{Bmatrix} \{X\} \\ \{Y\} \end{Bmatrix} \tag{15}$$

where I is an identity or unit matrix. The general solution of this set of equations results in eigenvalues λ with complex roots such that one can write $\lambda = \lambda_r \pm i\lambda_i$, where $i = \sqrt{-1}$. The positive and negative signs of the real part of the eigenvalue (λ_r) indicate the unstable and stable behaviors of the rotating system. In dynamic stability problems, the usual interest is in identifying the unstable zones, i.e., to determine the boundaries of the principal instability region in the frequency domain, where parametric resonance is most likely to occur. The boundary frequency equation (roots of Ω) for the dynamic system governed by the Mathieu-Hill equation, Eq. (6), is written as

$$\det. \begin{vmatrix} (K + \frac{1}{2}B - \frac{\Omega^2}{4}M) & -\frac{\Omega}{2}C \\ \frac{\Omega}{2}C & (K - \frac{1}{2}B - \frac{\Omega^2}{4}M) \end{vmatrix} = 0 \tag{16}$$

After some lengthy algebraic manipulations, Eq. (16) reduces to finding the eigenvalues of the following matrix:

$$\begin{bmatrix} 0 & 0 & 2I & 0 \\ 0 & 0 & 0 & 2I \\ M^{-1}(2K+B) & 0 & 0 & -2M^{-1}C \\ 0 & M^{-1}(2K-B) & 2M^{-1}C & 0 \end{bmatrix} \tag{17}$$

Since the matrices M, C, K, and B are fully defined and known, either of the complex eigenvalue problems described by Eqs. (15) and (17), respectively, are easily solved by using the IMSL-EISPACK routine.[14] The form in Eq. (15) is more suitable for determining the stability for a known value of the periodic load P_t and excitation frequency Ω. On the other hand, the form in Eq. (17) is more convenient at the design stage, where the attempt is to identify the instability zones a priori and, if possible, to avoid them altogether. After determining the roots of λ in Eq. (15), one can quickly determine the effective logarithmic decrement as well by

$$\text{logarithmic decrement} = -2\pi(\lambda_r / |\lambda_i|) \tag{18}$$

The dynamic system will be stable only when the effective logarithmic decrement for each mode is either zero or positive. The parametric resonance condition can be determined by the imaginary part (λ_i) of the eigenvalue. Simple parametric resonance in the neighborhood of twice the natural frequency of the system would always be observed. The imaginary roots of λ also yield a very important piece of data, i.e., the critical speeds of the rotor.[12]

Concluding Remarks

In this Note, we have derived all the equations necessary for determining the stability conditions of a uniform viscoelastic rotor-disk system subjected to a pulsating axial load. Although the mathematical derivations have been shown only for the first buckled mode, the method is general enough to analyze any of the higher modes as well. In addition, this method provides the advantage that it is not limited to diagonal forms of the M, C, and K matrices. It is not even necessary to decouple the equations of motion by the diagonalization process of orthogonal modes. Yet one can compute the effective value of logarithmic decrement for each mode separately.

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